JOURNAL OF COMPUTATIONAL PHYSICS 33, 102–117 (1979)

On the Numerical Solution of the Drift Wave Equations by Means of Invariant Imbedding*

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Received February 22, 1978; revised October 17, 1978

The method of invariant imbedding is applied to solve the system of differential equations for the collisionless drift wave in a sheared magnetic field in the case of finite β_i. Eigenvalues and eigenfunctions are found. Invariant imbedding is found to be fast and accurate, overcoming the precision problems inherent in the ordinary shooting method.

I. INTRODUCTION

Recently, the authors have shown numerically that the collisionless drift wave is stable. The results of our calculations for the electrostatic case as well as a comparison with the analytic results of Catto and Tsang are presented in Tsang, Catto, Whitson and Smith [6].

It is the purpose of the present article to describe the numerical methods used to calculate the eigenvalues and eigenfunctions of the drift wave equations, especially in the case of finite β_i.

We point out that the reason previous papers have concluded erroneously that the drift wave passes from a stable to an unstable state as $k_y \rho_i$ increases is the use of the term $i(\pi)^{1/2}(x_e/x)\exp(-x_e^2/x^2)$ to approximate $(x_e/x)\, Z(x_e/x)$ where Z represents the plasma dispersion function (see Fried and Conte, [1]). In fact the behavior of these two functions is quite different near $x = 0$.

To avoid the possibility of an error due to the incorrect approximation of some term, we use a rather involved set of equations to study the drift wave.

Our starting point, Tsang [5], is the system of ordinary differential equations for the drift wave in a finite-β inhomogeneous plasma in a sheared magnetic field

$$\frac{d^2E}{dx^2} - (g + \kappa_1)\, E = \frac{\nu}{x}\left(\Delta \frac{d^2B}{dx^2} + gB\right),$$
$$\frac{d^2B}{dx^2} - bB = -\frac{1}{x}\left(\frac{d^2E}{dx^2} - \kappa E\right). \tag{1}$$

† Prime contractor for the Department of Energy.

0021-9991/79/100102–16$02.00/0

The plasma density is assumed to vary in the x direction, E is the component of the perturbed electric field in a direction perpendicular to x, and B is the component of the perturbed magnetic field in a direction parallel to x. x is given in dimensionless units ($x = \xi/\rho_i$ where ξ is the actual length and ρ_i is the ion gyroradius) while the other quantities (also dimensionless) are given by the equations

$$
\begin{aligned}
g &= \lambda + M + \sigma_e ,\\
\lambda &= (\omega/\omega_* - 1)/D_3 ,\\
\kappa &= [(1 - \Gamma_0)(\tau\omega/\omega_* + 1) + \eta_i b(\Gamma_0 - \Gamma_1)]/D_2 ,\\
\kappa_1 &= D_2\kappa/D_3 ,\\
M &= [(\sigma_i\tau + \tau\omega/\omega_* + 1)\,\Gamma_0 + b\eta_i(\Gamma_1 - \Gamma_0)(1 + \xi_i Z(\xi_i))]/D_3 ,\\
\sigma_e &= [(\omega/\omega_* - 1 + \tfrac{1}{2}\eta_e)\,\xi_e Z(\xi_e) - \eta_e\xi_e^2(1 + \xi_e Z(\xi_e))]/D_3 ,\\
\sigma_i &= (\omega/\omega_* + (1 - \tfrac{1}{2}\eta_i)/\tau)\,\xi_i Z(\xi_i) + \eta_i\xi_i^2(1 + \xi_i Z(\xi_i))/\tau,\\
D_3 &= -\sigma_i\tau(\Gamma_0 - \Gamma_1) - \eta_i(\Gamma_0 - 2b(\Gamma_0 - \Gamma_1))\,\xi_i Z(\xi_i),\\
\Delta &= [(\tau\omega/\omega_* + 1 + \sigma_i\tau)(\Gamma_0 - \Gamma_1) - \eta_i(1 + \xi_i Z(\xi_i))(\Gamma_0 - 2b(\Gamma_0 - \Gamma_1))]/D_3 ,\\
D_2 &= (\tau\omega/\omega_* + 1)(\Gamma_0 - \Gamma_1) + \eta_i[\Gamma_0 - 2b(\Gamma_0 - \Gamma_1)],\\
\xi_e &= x_e/x,\\
\xi_i &= x_i/x,\\
x_i &= (\omega/\omega_*)(L_s/L_n)(\tau/\sqrt{2}),\\
x_e &= (\omega/\omega_*)(L_s/L_n)((m_e/M_i)(\tau/2))^{1/2},\\
\nu &= \beta_i\tau(\omega/\omega_*)\,D_2(L_s/L_n)^2,
\end{aligned}
$$

where $Z(\xi)$ is the plasma dispersion function, and

$$
\begin{aligned}
\Gamma_0 &= e^{-b}I_0(b),\\
\Gamma_1 &= e^{-b}I_1(b),
\end{aligned}
$$

for I_0 and I_1 the modified Bessel functions.

We observe that once the real constants η_i, η_e, β_i, $\tau\ (=T_e/T_i)$, $b\ (=k_y^2\rho_i^2)$, m_e/M_i, and L_s/L_n are given, the coefficients of (1) are all determined as functions of x and the complex parameter ω/ω_*.

The basic eigenvalue problem is then to find a value of the complex parameter ω/ω_* and complex solutions of (1), E, B, on the interval $0 \leqslant x < \infty$ which satisfy

$$E'(0) = 0 \qquad (E \text{ even}), \tag{3}$$

$$B(0) = 0 \qquad (B \text{ odd}), \tag{4}$$

$$E \text{ decays to } 0 \text{ as } x \to \infty, \tag{5}$$

$$B \text{ decays to } 0 \text{ as } x \to \infty. \tag{6}$$

It is clear that the coefficients of (1) depend on ω/ω_* in a nonlinear manner, and hence many of the standard numerical techniques for finding eigenvalues are not applicable.

A mode is stable if $\mathrm{Im}(\omega/\omega_*) < 0$, unstable if $\mathrm{Im}(\omega/\omega_*) > 0$, and marginally stable if $\mathrm{Im}(\omega/\omega_*) = 0$.

If we multiply the second equation (1) by $\nu\Delta/x$ and add to the first equation (1), we obtain an equation for E'' in terms of E and B. If we multiply the first equation (1) by $-1/x$ and add to the second equation (1), we obtain an equation for B'' in terms of E and B. The result is

$$\begin{bmatrix} E \\ B \end{bmatrix}'' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E \\ B \end{bmatrix}, \tag{7}$$

where

$$\begin{aligned} a_{11} &= (g + \kappa_1 + \nu\,\Delta\kappa/x^2)/h, \\ a_{12} &= \nu(\Delta b + g)/xh, \\ a_{21} &= (\kappa - g - \kappa_1)/xh, \\ a_{22} &= (b - \nu g/x^2)/h, \\ h &= 1 + \nu\Delta/x. \end{aligned} \tag{8}$$

Thus, we have

$$\begin{bmatrix} E \\ B \end{bmatrix}'' = A \begin{bmatrix} E \\ B \end{bmatrix}, \tag{9}$$

where

$$A = A(x, \omega/\omega_*\,; \eta_e, \eta_i, \beta_i, \tau, b, m_e/M_i, L_s/L_n). \tag{10}$$

In dealing with a single second order equation, the method of shooting backwards to the origin is quite adequate to obtain a solution. Thus the electrostatic case is amenable to such a procedure. On the other hand, if one wants to solve the full electromagnetic system (9), it is necessary to shoot backwards with two linearly independent vectors

$$\begin{bmatrix} E_1 \\ B_1 \\ E_1' \\ B_1' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_2 \\ B_2 \\ E_2' \\ B_2' \end{bmatrix}.$$

This procedure may be used to evaluate the determinant

$$\begin{vmatrix} E_1' & E_2' \\ B_1 & B_2 \end{vmatrix}$$

and set it equal to zero at the origin. (See Smith and Whitson [4]).

Unfortunately, if we start with two linearly independent vectors (let us say that they are even orthogonal) they may become almost parallel as we shoot towards the origin. This problem becomes more pronounced, the greater the shooting length. Calculating the determinant with two almost proportional columns leads to the precision problems observed in [4]. The lack of precision manifests itself by causing

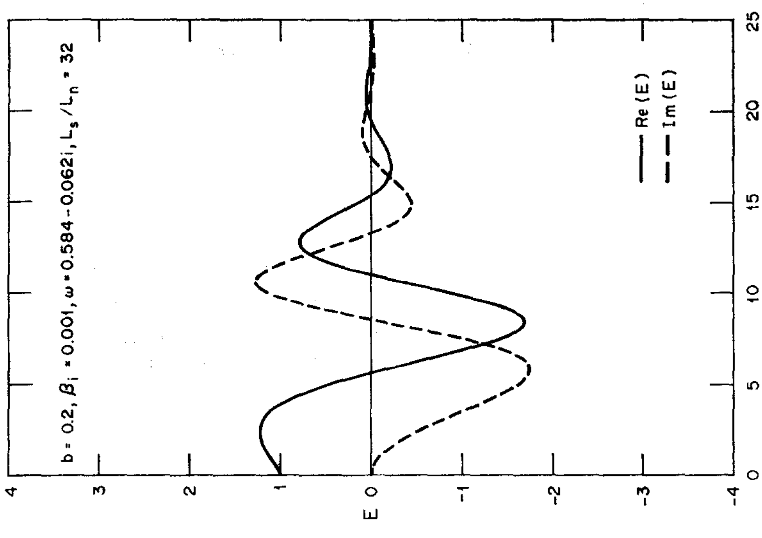

FIGURE 1

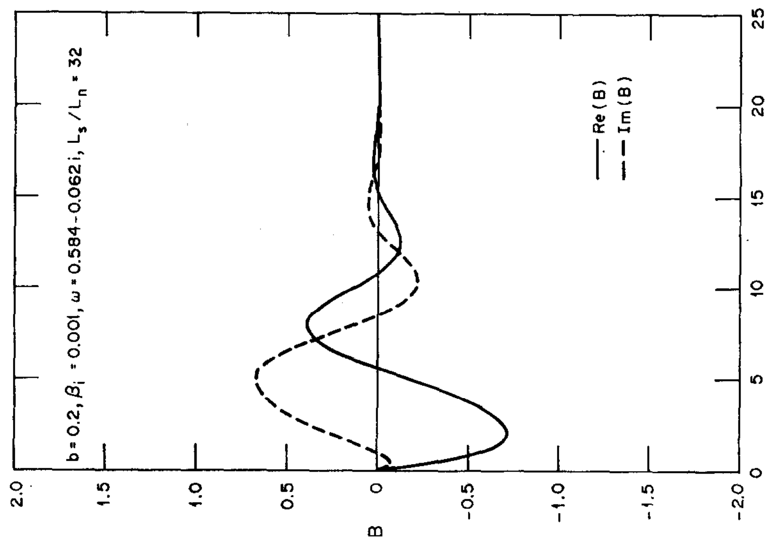

FIGURE 2

convergence failure in the root finder for increasing shooting length. There are several known methods for alleviating the above problem (see Guderley [7] and Davey [8]). They are: the method of orthonormalization which realigns the shooting vectors so they become almost orthogonal at fixed intervals, the method of parallel shooting which starts shooting from several different initial points, and the method of invariant imbedding which we have chosen to implement. To acquaint the reader with this method we briefly outline the procedure in the electrostatic case ($\beta_i = 0$).

II. The Electrostatic Case

If $\beta_i = 0$, then $\nu = 0$ and the system (7) reduces to

$$E'' = aE, \qquad \text{where} \qquad a = a_{11}, \tag{11}$$

together with the boundary conditions

$$E'(0) = 0, \qquad E(\infty) = 0. \tag{12}$$

We now define $s = E'/E$ and $r = E/E' = 1/s$. This yields

$$s' = -s^2 + a, \tag{13}$$

$$r' = -ar^2 + 1 \tag{14}$$

with the boundary conditions

$$s(0) = 0, \qquad \operatorname{Re} r < 0 \quad \text{as} \quad x \to \infty, \tag{15}$$

and the switching conditions

$$r = \frac{1}{s}, \qquad s = \frac{1}{r}. \tag{16}$$

If one starts with the initial condition $s = 0$ at $x = 0$ and integrates in the direction of increasing x, switching to r when $|s| > 1$, it is found that, as x increases beyond some critical length, the solution always turns to $\operatorname{Re} r > 0$ (E growing). This is caused by an inherent instability in the direction of increasing x for that solution of (14) which satisfies $\operatorname{Re} r < 0$. For this reason, we start our solution of (14) at some large value of x, say $x = l$, with the initial condition

$$r = 0 \quad \text{at} \quad x = l \tag{17}$$

or the initial condition

$$r = 1/(a^{1/2} - a'/4a) \quad \text{at} \quad x = l, \tag{18}$$

the square root in (18) being chosen so that $\operatorname{Re} r < 0$. This is of course the WKB approximation. The solution to (14) is then followed for decreasing x until $|r| > 1$ when we switch to s and equation (13), which we continue to follow until x decreases to zero. In contrast to the previous method, this method of "shooting backwards" to the origin is highly stable. The complete procedure for finding an eigenvalue may now be described. First, a value of ω/ω_* is chosen. With this value of ω/ω_*, (13) and (14) are shot backwards to the origin. This yields a value of s at $x = 0$. We denote this value of s by $f(\omega/\omega_*)$. Thus, the process of solving the initial value problem may be considered as a means for evaluating the function $f(\omega/\omega_*)$. Conditions (15) require that we obtain a value of ω/ω_* such that $f(\omega/\omega_*) = 0$.

We found the complex secant method adequate for our purposes. That is, we selected an initial guess $(\omega/\omega_*)_0$, shot to the origin to find $f((\omega/\omega_*)_0)$, then selected a second guess $(\omega/\omega_*)_1 = (1 \pm t)(\omega/\omega_*)_0$ for t a small number and shot in to find $f((\omega/\omega_*)_1)$. Further guesses were then found successively by the formula

$$(\omega/\omega_*)_n = (\omega/\omega_*)_{n-1} - \frac{f((\omega/\omega_*)_{n-1})((\omega/\omega_*)_{n-1} - (\omega/\omega_*)_{n-2})}{f((\omega/\omega_*)_{n-1} - f((\omega/\omega_*)_{n-2})}. \tag{19}$$

In practice, rapid convergence (from 5–15 iterations) was obtained for six-place accuracy in ω/ω_* if one started with a good guess.

We observe that solutions to the eigenvalue problem using (17) and (18) as initial conditions agree for large enough starting lengths l. Equation (18) permits the use of shorter lengths l without loss of accuracy. Nevertheless, we have found that the procedure should be checked occasionally with large shooting lengths l since the value of l needed for a given accuracy criterion varies with respect to the input parameters.

To recover our eigenfunctions in the invariant imbedding procedure, we observe that in an interval where r is calculated we have

$$\begin{aligned} E &= rE', \\ E'' &= aE = arE'. \end{aligned} \tag{20}$$

Thus, letting $m = E'$ we have

$$m' = arm,$$

a first-order equation which may be initialized by

$$m(l) = \epsilon.$$

Because m grows as we shoot inward, it is advantageous to calculate $q = 1/m$, which decays. We find

$$q' = -arq, \tag{21}$$

$$q(l) = \frac{1}{\epsilon}. \tag{22}$$

From q we have

$$E' = \frac{1}{q} \quad \text{and} \quad E = rE' = \frac{r}{q}. \tag{24}$$

In the s-interval we have

$$E' = sE. \tag{25}$$

Again we compute with

$$k = \frac{1}{E},$$

so that

$$k' = -sk. \tag{26}$$

From k we have

$$E = \frac{1}{k}, \qquad E' = \frac{s}{k}, \tag{27}$$

the initial value for k being obtained from the formula $k = q/r$.

In switching from s to r, we have

$$q = \frac{k}{s}. \tag{28}$$

An extra shot may be used to normalize taking

$$q(l) = \frac{1}{\epsilon k(0)}. \tag{29}$$

III. The Full System of Equations

The basic program of imbedding for systems is given in Scott [2, Chap. VIII].

Just as in a shooting code for two dependent variables, we need two linearly independent solutions

$$\begin{bmatrix} E_1 \\ B_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$$

to (7).

The solutions may be combined to give the matrix equation

$$\begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}'' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}. \tag{30}$$

In analogy to the quantity r introduced for the single equation (11), we introduce the matrix quantity R defined by

$$\begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} = R \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}. \tag{31}$$

This yields

$$\begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} = R' \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} + R \begin{bmatrix} E_1'' & E_2'' \\ B_1'' & B_2'' \end{bmatrix},$$

or from (30)

$$R = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}, \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$\begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} = R' \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} + RA \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}$$

$$= R' \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} + RAR \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}.$$

Thus, if we assume $\begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}$ to be nonsingular, we obtain

$$R' = I - RAR, \tag{32}$$

which is a first-order system of equations in four complex unknowns r_{ij}, the components of the matrix R. To fit the boundary conditions (3) and (4) we must use a matrix S which is not the inverse of R.

In fact, if we define $E' = u$ and $B' = v$, we may rewrite the system (7) in the form

$$\begin{aligned} E' &= u \\ u' &= a_{11}E + a_{12}B \\ B' &= v \\ v' &= a_{21}E + a_{22}B, \end{aligned}$$

or

$$\begin{bmatrix} u \\ B \end{bmatrix}' = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ B \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ v \end{bmatrix}$$

$$\begin{bmatrix} E \\ v \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} u \\ B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E \\ v \end{bmatrix}.$$

It follows that

$$\begin{aligned} \begin{bmatrix} E' \\ B \end{bmatrix}' &= \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E' \\ B \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ B' \end{bmatrix} \\ \begin{bmatrix} E \\ B' \end{bmatrix}' &= \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} E' \\ B \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E \\ B' \end{bmatrix} \end{aligned} \tag{33}$$

We may then introduce the matrix S by the formula

$$\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = S \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix} \tag{34}$$

where $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$ are the same independent solutions given before. Thus

$$\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix}' = S' \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix} + S \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}'.$$

Using (33), we find that

$$\begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}$$
$$= S' \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix} + S \left\{ \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix} \right\}$$

Employing (34), we obtain

$$\begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} S \begin{bmatrix} E_1 & E_1 \\ B_1' & B_2' \end{bmatrix} + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}$$
$$= S' \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix} + S \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix} + S \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}.$$

Assuming that $\begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}$ is nonsingular, we find

$$S' = -S \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S - S \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix} + \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix} S + \begin{bmatrix} a_{11} & 0 \\ 0 & 1 \end{bmatrix}, \tag{35}$$

which is again a first-order system in the four unknowns s_{ij} of the matrix S. Relations may be obtained which relate the elements of R and S as follows:

We have from (31) and (34)

$$\begin{bmatrix} E_i' \\ B_i \end{bmatrix} = S \begin{bmatrix} E_i \\ B_i' \end{bmatrix}, \qquad i = 1, 2,$$
$$\begin{bmatrix} E_i \\ B_i \end{bmatrix} = R \begin{bmatrix} E_i' \\ B_i' \end{bmatrix}.$$

Thus

$$E_i' = s_{11} E_i + s_{12} B_i',$$
$$B_i = s_{21} E_i + s_{22} B_i',$$
$$E_i = r_{11} E_i' + r_{12} B_i',$$
$$B_i = r_{21} E_i' + r_{22} B_i'.$$

Solving the last two equations for E_i' and B_i, we have

$$E_i' = \frac{1}{r_{11}} E_i - \frac{r_{12}}{r_{11}} B_i',$$

$$B_i = r_{21}\left(\frac{1}{r_{11}} E_i - \frac{r_{12}}{r_{11}} B_i'\right) + r_{22}B_i',$$

$$B_i = \frac{r_{21}}{r_{11}} E_i + \left(r_{22} - \frac{r_{12}r_{21}}{r_{11}}\right) B_i'.$$

Hence, the connecting equations are

$$s_{11} = \frac{1}{r_{11}}, \qquad s_{12} = \frac{-r_{12}}{r_{11}},$$
$$s_{21} = \frac{r_{21}}{r_{11}}, \qquad s_{22} = r_{22} - \frac{r_{12}r_{21}}{r_{11}}, \tag{36}$$

which may be solved backwards to yield

$$r_{11} = \frac{1}{s_{11}}, \qquad r_{12} = \frac{-s_{12}}{s_{11}},$$
$$r_{21} = \frac{s_{21}}{s_{11}}, \qquad r_{22} = s_{22} - \frac{s_{12}s_{21}}{s_{11}}. \tag{37}$$

The procedure we use is similar to that employed for the single equation. Since E and B vanish as $x \to \infty$, we may take $E_i = 0$ and $B_i = 0$ at $x = l$, $i = 1, 2$. This yields the initial condition

$$r_{11} = 0, \qquad r_{12} = 0, \qquad r_{21} = 0, \qquad r_{22} = 0 \quad \text{at} \quad x = l. \tag{38}$$

The condition (38) is analogous to the condition (17) for a single equation. A WKB condition similar to (18) for the single equation may be derived, but the form is much more involved than (18) since it requires the square root of a matrix. So far we have had adequate success with (38). We use (38) to start the backwards shot on R at some large length l. The values of R are then followed until a value of x at which $|r_{11}| > 1$, when the code switches and S is followed until a small value of x, $x = \epsilon_0$, is reached. We cannot shoot all the way to $x = 0$ since the coefficients in equation (35) are singular at $x = 0$. This may be seen from (8). In practice we take $\epsilon_0 = 10^{-6}$. As we have seen, the desired solution E, B must satisfy

$$E = c_1E_1 + c_2E_2,$$
$$B = c_1B_1 + c_2B_2.$$

Thus, if we are to satisfy the conditions $E' = 0$, $B = 0$ at $x = \epsilon_0$, we must have

$$c_1 E_1' + c_2 E_2' = 0,$$
$$c_1 B_1 + c_2 B_2 = 0$$

at $x = \epsilon_0$. This implies that at $x = \epsilon_0$

$$\det \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = 0$$

if we are to have nontrivial solutions. The last condition requires that $\det S = 0$ at $x = \epsilon_0$ since the other matrix on the right hand side of (34) is nonsingular. Hence, we may let $f(\omega/\omega_*)$ be set equal to the value of $\det S$ at $x = \epsilon_0$ and proceed as in the case of the scalar $r - s$ equations. That is, we use a root finder to determine a value of ω/ω_* for which $f(\omega/\omega_*) = 0$.

IV. Recovery of the Eigenfunctions

The solutions E and B may be recovered from the imbedding equations once we have determined an eigenvalue ω/ω_*. This may be done as follows. First we find $\begin{bmatrix} E_1 \\ B_1 \end{bmatrix}$ and $\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$, linearly independent solutions of (7), then

$$\begin{bmatrix} E \\ B \end{bmatrix} = c_1 \begin{bmatrix} E_1 \\ B_1 \end{bmatrix} + c_2 \begin{bmatrix} E_2 \\ B_2 \end{bmatrix}.$$

From (31) we have

$$\begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} = R \begin{bmatrix} E_1' & E_1' \\ B_1' & B_2' \end{bmatrix} \tag{39}$$

and from (30) we have

$$\begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}' = A \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix} \tag{40}$$

Therefore

$$\begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}' = AR \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix}.$$

Thus, the matrix

$$M = \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} \tag{41}$$

satisfies the first-order system

$$M' = ARM. \tag{42}$$

We may choose initial conditions so that

$$\begin{bmatrix} E_1' \\ B_1' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} E_2' \\ B_2' \end{bmatrix}$$

are independent, e.g.

$$E_1'(l) = \epsilon_1, \qquad B_1'(l) = 0, \qquad E_2'(l) = 0,$$
$$B_2'(l) = \epsilon_2, \qquad \epsilon_1, \epsilon_2 \neq 0.$$

Hence, M satisfies the initial conditions

$$M(l) = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix} \tag{43}$$

Numerically, the solution M grows as we shoot from l inward so that we may with advantage solve instead for Q, the inverse of M.

We have

$$QM = I,$$
$$Q'M + QM' = 0,$$
$$Q'M + QARM = 0,$$

and, since M is nonsingular,

$$Q' = -QAR. \tag{44}$$

The sign in the equation for Q has been reversed, and we are thus solving for decaying modes. From (43) we have the initial condition

$$Q(l) = \begin{bmatrix} \epsilon_1^{-1} & 0 \\ 0 & \epsilon_2^{-1} \end{bmatrix}. \tag{45}$$

Thus, we solve equation (44) for Q subject to (45). We then make use of

$$M = Q^{-1}, \tag{46}$$

and finally defining

$$N = \begin{bmatrix} E_1 & E_2 \\ B_1 & B_2 \end{bmatrix}, \tag{47}$$

we may calculate from (39)

$$N = RM. \tag{48}$$

Therefore, in the regime where R is calculated we know the values of $[{E_1 \atop B_1}]$ and $[{E_2 \atop B_2}]$. In the regime where S is calculated, we have

$$\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = S \begin{bmatrix} E_1 & E \\ B_1' & B_2' \end{bmatrix}, \tag{49}$$

as well as

$$\begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix}\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix}\begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}.$$

It follows that

$$\begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}' = \begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix}\begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}.$$

Hence, if we define

$$L = \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix}, \tag{50}$$

we have

$$L' = \left\{\begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix}\right\} L. \tag{51}$$

Again, defining $PL = I$

$$P'L + PL' = 0,$$

$$P'L + P\left\{\begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix}\right\} L = 0, \tag{52}$$

$$P' = -P\left\{\begin{bmatrix} 1 & 0 \\ 0 & a_{22} \end{bmatrix} S + \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix}\right\}.$$

We thus calculate P and then

$$L = P^{-1}. \tag{53}$$

Finally, if we define

$$K = \begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix}, \tag{54}$$

we have

$$K = SL. \tag{55}$$

Thus, in K and L we have all the quantities E_1, E_2, B_1, B_2, E_1', E_2', B_1', and B_2'. The initial conditions for P may be determined from the values of these same quantities in the R regime as follows:

$$L = \begin{bmatrix} E_1 & E_2 \\ B_1' & B_2' \end{bmatrix} = \begin{bmatrix} N_{11} & N_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{56}$$

$$P = L^{-1}. \tag{57}$$

A switch back to R may be accomplished by

$$M = \begin{bmatrix} E_1' & E_2' \\ B_1' & B_2' \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ L_{21} & L_{22} \end{bmatrix}, \tag{58}$$

which yields initial conditions for $Q = M^{-1}$.

One shot inward is then used to evaluate $E_1(\epsilon_0)$, $E_2(\epsilon_0)$, $B_1(\epsilon_0)$, $B_2(\epsilon_0)$, $E_1'(\epsilon_0)$, and $E_2'(\epsilon_0)$. Now if $\begin{bmatrix} E \\ B \end{bmatrix} = c_1\begin{bmatrix} E_1 \\ B_1 \end{bmatrix} + c_2\begin{bmatrix} E_2 \\ B_2 \end{bmatrix}$,

$$\begin{aligned} E &= c_1E_1 + c_2E_2 , \\ B &= c_1B_1 + c_2B_2 , \\ E' &= c_1E_1' + c_2E_2' , \qquad \text{and} \\ B' &= c_1B_1' + c_2B_2' . \end{aligned} \tag{59}$$

Hence we may set $E'(\epsilon_0)$ and $B(\epsilon_0) = 0$, which yields

$$\begin{aligned} c_1E_1'(\epsilon_0) + c_2E_2'(\epsilon_0) &= 0, \\ c_1B_1(\epsilon_0) + c_2B_2(\epsilon_0) &= 0. \end{aligned} \tag{60}$$

We have already arranged for an ω/ω_* to make $\det S = 0$ at $x = \epsilon_0$ so that $\det\begin{bmatrix} E_1' & E_2' \\ B_1 & B_2 \end{bmatrix} = 0$. Thus, we need only select

$$\begin{aligned} c_1 &= -cB_2(\epsilon_0), \\ c_2 &= cB_1(\epsilon_0) \end{aligned} \tag{61}$$

and, if we normalize to $E(\epsilon_0) = 1$, we have

$$\begin{aligned} 1 &= E(\epsilon_0) = c_1E_1(\epsilon_0) + c_2E_2(\epsilon_0), \\ 1 &= -cB_2(\epsilon_0)\, E_1(\epsilon_0) + cB_1(\epsilon_0)\, E_2(\epsilon_0), \end{aligned}$$

so that

$$\begin{aligned} c &= \frac{1}{B_1(\epsilon_0)\, E_2(\epsilon_0) - B_2(\epsilon_0)\, E_1(\epsilon_0)}, \\ c_1 &= \frac{-B_2(\epsilon_0)}{B_1(\epsilon_0)\, E_2(\epsilon_0) - B_2(\epsilon_0)\, E_1(\epsilon_0)}, \\ c_2 &= \frac{B_1(\epsilon_0)}{B_1(\epsilon_0)\, E_2(\epsilon_0) - B_2(\epsilon_0)\, E_1(\epsilon_0)}, \end{aligned}$$

or written in terms of K and L

$$\begin{aligned} c_1 &= \frac{-K_{22}(\epsilon_0)}{K_{21}(\epsilon_0)\, L_{12}(\epsilon_0) - K_{22}(\epsilon_0)\, L_{11}(\epsilon_0)}, \\ c_2 &= \frac{K_{21}(\epsilon_0)}{K_{21}(\epsilon_0)\, L_{12}(\epsilon_0) - K_{22}(\epsilon_0)\, L_{11}(\epsilon_0)}; \end{aligned} \tag{62}$$

If we write (59) in matrix form, we have

$$\begin{bmatrix} E \\ B \end{bmatrix} = N \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad \begin{bmatrix} E' \\ B' \end{bmatrix} = M \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{63}$$

or alternatively

$$\begin{bmatrix} E \\ B' \end{bmatrix} = L \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad \begin{bmatrix} E' \\ B \end{bmatrix} = K \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{64}$$

Thus, after a first shot which determines c_1 and c_2, we may use (63) and (64) on a second shot to determine the fields E, B, E', B' in the appropriate regime.

V. Numerical Procedures

We used the value $\omega/\omega_* \sim 1$ as a guess for the principal mode in the case $\eta_e = \eta_i = \beta_i = b = 0$. The various parameters appearing in the equation were then turned on slowly, using the value of ω/ω_* for a given parameter value as a guess for ω/ω_* at the new parameter value.

To achieve a high level of accuracy, we used the routine DE of Shampine and Gordon [3] to compute solutions to the initial value problem. We remark that the odd modes ($E = 0$, $B' = 0$ at $x = 0$) may be easily found with only slight coding changes by appropriately interchanging the roles of E and B in the code.

We include some sample eigenmodes for the case of finite β_i.

Acknowledgments

The authors would like to thank J. D. Callen, P. J. Catto, and K. T. Tsang for their encouragement and interest in our numerical work on these equations.

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